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Singular behaviour of the free energy for random Ising chains

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Abstract. The Ising chains in a random external field or with random interactions are studied. The equivalence of these models is shown, and the functional equation for the quenched free energy is derived. For the system in a random external field we prove rigorously the non-analytical behaviour of the free energy in the limit of large constant interactions: $f(J) \approx \exp(-2J\alpha^*)$, with positive Derrida–Hilhorst roots $\alpha^* < 1$ determined by $\langle \exp(-2H\alpha^*) \rangle = 1$. The generalization to quasi-one dimensional systems is given. The system with random interactions, the fraction of which is infinitely large, is solved exactly. For large, but finite interactions, we prove again that the singular behaviour of f can be found, depending on the magnitude of the external magnetic field and on the probability distribution of interactions.

1. Introduction

Low-dimensional systems, in spite of their restricted applicability, correctly describe a number of phenomena associated with higher dimensions. Their equilibrium statistics can usually be determined exactly and so they may be used as a test for general hypotheses or make some new suggestions. However, once the model becomes random, the space variation of its parameters prevents expression of the free energy in a closed form, except in some special examples.

In the theory of the random-field Ising chain with constant (dimensionless) interactions J (model I; for a recent review see [1]) and the Ising chain with random interactions in a uniform (dimensionless) field H^* (model II), the closed-form solutions have been obtained at zero temperature [2–4]; building the model's random parameters from the known solution, at a specific temperature [5]; considering particular non-trivial distributions of random fields [6, 7] or taking the fields on the fraction of sites infinitely strong [8, 9], at arbitrary temperatures.

When the exact results are not available, one combines analytical and numerical methods [10] or develops series expansions of chosen quantities in powers of some small parameter. For model I, Derrida and Hilhorst [9] elaborated a large-J expansion of the quenched free energy in powers of $a = \exp(-2J)$. They first derived the integral equation for the unique probability distribution P of auxiliary recursion variables [4] and then averaged the free energy f_a. In the limit of infinite interactions $a \to 0$ they showed that f_a is analytical in the lowest order of a only if the (site-independent) mean value of field variables $z_n = \exp(-2H_n)$ fulfils the condition

$$\langle z \rangle < 1 . \tag{1}$$

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If the mean value of z is higher than one, they obtained the singular behaviour $f_a \approx a^{2\alpha^*}$ with the exponent α^* being given by the positive root of $\langle z^{\alpha^*} \rangle = 1$. The procedure of deriving Derrida–Hilhorst roots has not been justified rigorously, but the conclusion turns out to be valid for a wide class of distributions of z [7,9]. For model II, the non-analyticity has been observed for the ground state magnetization in the limit of a vanishing external field [11,12].

The framework in which we set the main results of this paper is presented in sections 2 and 3. In section 2, we map models I and II one into another. On the basis of this 'symmetry' we obtain the *one* integral equation for function $f_a(b)$ ($a = \exp(-2J)$, $\tanh(H^*)$ for models I and II, respectively), which for the value of the formal parameter $b = a$ determines the free energy f_a of systems (section 3). We use this equation, which slightly differs from that derived in [6,7], as a starting point for the study of both models.

The existence of the equation for the free energy rather than for the probability distribution P is of special importance. We believe that it enables us to study both models in a much simpler way, especially in asymptotic regimes with 'almost infinite interactions'. The probability distribution P can be solved exacly in the limit of infinite constant or (judiciously chosen) random interactions for models I and II, respectively. Its form, however, considerably changes when interactions slightly differ from infinity. On the other hand, f_a should change only slightly in this process and so the proposed formulation could be more suitable.

As an example of our treatment, we re-analyse model I in section 3. We explain the origin of Derrida–Hilhorst roots and reproduce the singular behaviour of f_a in a rigorous and simpler way compared with [9].

Section 4 brings an application of the formalism to the quasi-one-dimensional case. The integral equation for f_a can be also written but in space of $L = 2^M - 1$ formal parameters $b_1, \ldots, b_L$ for a strip of width M. We show that in such systems the condition of the analyticity of f_a coincides with (1). If condition (1) does not valid, the breakdown of the Taylor expansion indicates the non-analyticity of f_a in the $2M$th order of a.

Section 5 deals with a counterpart of the above mentioned random-field Ising chain in models II with a special choice of random bonds J_n^*: $\tanh J_n^* = \pm K$ and 0 with corresponding probabilities. In the limit of infinite random interactions ($K = 1$) we solve this model exactly. When $K = 1 - \varepsilon$ ($\varepsilon \to 0$), we prove that under certain conditions f_a becomes non-analytical in the first, second, ..., nth order of the expansion in ε.

2. Dual properties of inhomogeneous Ising chains with broken symmetry

The ordinary duality transformation (for a review, see [13]) connects dual models possessing a global or local $Z(q)$ symmetry in dimensions higher than one. In this section we construct a duality transformation for the one-dimensional Ising model which $Z(2)$ symmetry is broken by an applied field. This duality generates a path between the Ising chain with diagonal inhomogeneity and that with non-diagonal inhomogeneity.

Let us first consider the Ising chain with constant nearest-neighbour interactions

of strength J in a varying field $\{H_n\}_{n=1}^N$. Its Hamiltonian is written as

$$\mathcal{H}(s) = -J \sum_{n=1}^{N-1} s_n s_{n+1} - \sum_{n=1}^{N} H_n s_n \tag{2}$$

where $s_n = \pm 1$ denotes the Ising spin variable at site n. The thermodynamic limit of the free energy per site, $f(J, \{H_n\})$, is given by

$$-f(J, \{H_n\}) = \lim_{N \to \infty} \frac{1}{N} \log Z^{(N)}(J, \{H_n\}) . \tag{3a}$$

Here

$$Z^{(N)}(J, \{H_n\}) = \sum_{\{s\}} \exp[-\mathcal{H}(s)] \tag{3b}$$

is the partition function with the summation going over all possible spin configurations. We note that J, $\{H_n\}$, $\mathcal{H}$, $f(J, \{H_n\})$ include the inverse temperature and so they are dimensionless.

The above theory does not possess a global $Z(2)$ symmetry because its Hamiltonian (2) is not invariant under a change in the sign of all spins s_n. This deficiency seems to prevent the construction of the duality transformation. However, the duality connects the partition functions of the dual models, so if we find a $Z(2)$ symmetric theory which partition function is identical, or proportional, to $Z^{(N)}(J, \{H_n\})$, we can use this theory in performing the duality transformation for the original model (2). With this aim, we take advantage of the invariance of $Z^{(N)}(J, \{H_n\})$ under the field-reversal transformation $H_n \to -H_n$ for all n and rewrite (3*b*) as

$$Z^{(N)}(J, \{H_n\}) = \frac{1}{2} \sum_{\{s, s_x\}} \exp[-\mathcal{H}(s, s_x)] \tag{4a}$$

where

$$\mathcal{H}(s, s_x) = -J \sum_{n=1}^{N-1} s_n s_{n+1} - \sum_{n=1}^{N} H_n s_n s_x \tag{4b}$$

is the Hamiltonian of the $Z(2)$ symmetric theory we are looking for. The introduced 'ghost' spin $s_x = \pm 1$ is placed on a new site x which does not lie on the considered chain.

The ghost spin concept allows us to transform the original problem of the Ising chain in an external field to the problem of interacting Ising spins on a two-dimensional structure O (figure 1) in zero field. We now can map this $Z(2)$ invariant model into its dual one. The mapping consists of a geometrical and an algebraic part: *the dual lattice* D is constructed from the original one O by placing a vertex in the centre of each elementary triangle of O and then connecting each vertex of D with its nearest neighbours as well as the dual point x situated outside the structure O (see figure 1); the Hamiltonian of *the dual Ising model* on D is given by

$$\mathcal{H}(\sigma) = -\sum_{n=1}^{N-1} J_n^* \sigma_n \sigma_{n+1} - H^* \sum_{n=1}^{N} \sigma_n \sigma_x \tag{5}$$

where σ_n (σ_x) $= \pm 1$ is a disorder variable associated with site n (x) of D. Let the crossing 'interactions' of the dual models be connected by the local duality relations

$$\exp(-2J) = \tanh(H^*) \tag{6a}$$

$$\exp(-2H_n) = \tanh(J_n^*)\,. \tag{6b}$$

Then the dual partition functions are proportional to each other:

$$Z^{(N)}(J,\{H_n\}) = (\sinh(2J))^{N/2} \prod_{n=1}^{N} (\sinh(2H_n))^{1/2}$$

$$\times \frac{1}{2} \sum_{\{\sigma,\sigma_x\}} \exp\left(\sum_{n=1}^{N-1} J_n^* \sigma_n \sigma_{n+1} + H^* \sum_{n=1}^{N} \sigma_n \sigma_x \right). \tag{7}$$

The inverse realization of the ghost spin method transforms the summation over spin configurations on the right-hand side of (7) to the statistical sum, $Z^{(N)}(\{J_n^*\}, H^*)$, of the Ising model inhomogeneous in two-spin interactions $\{J_n^*\}$ in the uniform field H^* with the Hamiltonian

$$\mathcal{H}(\sigma) = -\sum_{n=1}^{N-1} J_n^* \sigma_n \sigma_{n+1} - H^* \sum_{n=1}^{N} \sigma_n\,. \tag{8}$$

The final duality formula

$$Z^{(N)}(J,\{H_n\}) = (\sinh(2J))^{N/2} \prod_{n=1}^{N} (\sinh(2H_n))^{1/2} Z^{(N)}(\{J_n^*\}, H^*) \tag{9}$$

together with (6*a*) and (6*b*), produces a path between the two inhomogeneous Ising chains. It suitably enriches the concept of duality between the random-field Ising chain and the random-bond Ising models (in zero field) on a two-layer strip as presented in [14].

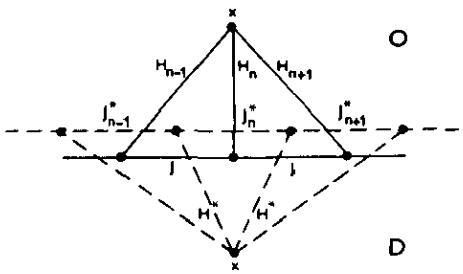

Figure 1. Dual inhomogeneous Ising chains in the ghost spin method: crossing interactions and fields are connected according to the duality relations (6*a*) and (6*b*).

All that has been said in this section can be generalized to all higher-dimensional Ising and gauge models which $Z(q)$ ($q = 2,3,\ldots$) symmetry is broken by an applied field.

In order to make use of the duality formula (9), we shall express the free energy per spin of the Ising chain in a varying field (equations (3*a*) and (3*b*)) in terms of non-linear recursive relations. In the spirit of the ordinary transfer-matrix technique we start from defining the quantity $Z_i^{(n)}$ ($n = 1,2,\ldots,N$; $i = 0,1$) which corresponds to the partition function of the chain fragment starting at site 1, ending at site n, with the fixed up (down) edge spin for $i = 0(1)$. It is easy to show that $Z_i^{(n)}$ obey the recursive relations

$$Z_0^{(n)} = \exp(H_n)[\exp(J)Z_0^{(n-1)} + \exp(-J)Z_1^{(n-1)}] \tag{10a}$$

$$Z_1^{(n)} = \exp(-H_n)[\exp(-J)Z_0^{(n-1)} + \exp(J)Z_1^{(n-1)}]. \tag{10b}$$

We propose new auxiliary variables $x_n = Z_1^{(n)}/Z_0^{(n)}$. They satisfy the recursive relation

$$x_n = \exp(-2H_n)\frac{\exp(-2J) + x_{n-1}}{1 + \exp(-2J)x_{n-1}}. \tag{11}$$

The free energy per spin is expressible in terms of x_n in the following way:

$$\begin{aligned} -f(J,\{H_n\}) &= \lim_{N\to\infty}\frac{1}{N}\log[Z_0^{(N)} + Z_1^{(N)}] \\ &= \lim_{N\to\infty}\frac{1}{N}\left[\sum_{n=2}^{N}\log\left(\frac{Z_0^{(n)} + Z_1^{(n)}}{Z_0^{(n-1)} + Z_1^{(n-1)}}\right) + \log(Z_0^{(1)} + Z_1^{(1)})\right] \\ &\approx \lim_{N\to\infty}\frac{1}{N}\left[\sum_{n=2}^{N}\log\left(\frac{1 + x_n}{1 + x_{n-1}}\right) + \sum_{n=2}^{N}\log\left(\frac{Z_0^{(n)}}{Z_0^{(n-1)}}\right)\right] \\ &\approx J + \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}H_n + \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log(1 + \exp(-2J)x_n). \end{aligned} \tag{12}$$

Here, we have neglected all terms vanishing in the limit $N \to \infty$.

The duality relations (6*a*), (6*b*) and (9) now allow us to express the free energies per site $f(J,\{H_n\})$ and $f(\{J_n^*\},H^*)$ through the only type of recurrence. Namely, let us define

$$f_a^{(N)}(\{z_n\}) = \frac{1}{N}\sum_{n=1}^{N}\log(1 + ax_n) \tag{13a}$$

with x_n given by

$$x_n = z_n\frac{a + x_{n-1}}{1 + ax_{n-1}} \tag{13b}$$

and $a = \exp(-2J)$ $(\tanh H^*)$, $z_n = \exp(-2H_n)$ $(\tanh J_n^*)$ for the Ising chain inhomogeneous in a field (two-spin interactions). Then, one finds

$$-f(J,\{H_n\}) = J + \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}H_n + \lim_{N\to\infty}f_{\exp(-2J)}^{(N)}(\{\exp(-2H_n)\}) \tag{14a}$$

$$-f(\{J_n^*\},H^*) = \log 2 + \log(\cosh(H^*)) + \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}\log(\cosh(J_n^*)) + \lim_{N\to\infty} f^{(N)}_{\tanh H^*}(\{\tanh J_n^*\}). \tag{14b}$$

Note that the values of parameters a, $\{z_n\}$ are restricted to the interval $\mathcal{D} = (0,\infty)$ for the Ising chain that is inhomogeneous in a field, and $\mathcal{D} = (-1,1)$ for the Ising chain that is inhomogeneous in interactions.

3. Random Ising chains

Let varying H_n in (14a) or J_n^* in (14b) be independent random variables. The problem of the quenched averaging of the corresponding free energies per site reduces to the problem of the quenched averaging of $f_a^{(N)}$ (equations (13a) and (13b)) for a given probability distribution $\rho(z)$ of z_n. The usual approach, initiated in [4] and developed by Derrida and co-workers (see e.g. [9, 11]), is based on the assumption that with increasing n a stationary probability distribution $P(x)$ of $x_n \in \mathcal{D}$, independent on n exists. Then

$$f_a = \lim_{N\to\infty} f_a^{(N)} = \int_{\mathcal{D}} \mathrm{d}x\, P(x)\log(1+ax) \tag{15}$$

where $P(x)$ is the solution of an integral equation deduced from recursion (13b). Here, we propose an approach which avoids the computation of $P(x)$ and directly leads to a functional equation for f_a itself. The method removes serious problems in determining $P(x)$ when the last reflects the non-trivial behaviour of f_a.

The method is based on the identity

$$f_a^{(N)} = \int_{\mathcal{D}}\cdots\int_{\mathcal{D}} \mathrm{d}z_1\cdots\mathrm{d}z_N\,\rho(z_1)\cdots\rho(z_N)\log(1+ax_N) \tag{16a}$$

where x_N (being the function of $z_1,\ldots,z_N$) is related to x_{N-1} (being the function of $z_1,\ldots,z_{N-1}$) by

$$x_N = z_N\frac{a+x_{N-1}}{1+ax_{N-1}}. \tag{16b}$$

Let us extend the definition of $f_a^{(N)}$ (equation (16a)) as follows:

$$f_a^{(N)}(b) = \int_{\mathcal{D}}\cdots\int_{\mathcal{D}} \mathrm{d}z_1\cdots\mathrm{d}z_N\,\rho(z_1)\cdots\rho(z_N)\log(1+bx_N). \tag{17}$$

It is clear that $f_a^{(N)} = f_a^{(N)}(a)$. Substituting x_N (16b) into (17) and after some simple algebra we obtain

$$f_a^{(N)}(b) = \langle\log(1+abz)\rangle + \left\langle f_a^{(N-1)}\left(\frac{a+bz}{1+abz}\right) - f_a^{(N-1)}(a)\right\rangle. \tag{18}$$

Here, $\langle\cdots\rangle$ denotes the averaging over $\rho(z)$. Since in the limit $N \to \infty$ $f_a^{(N)}(b) = f_a^{(N-1)}(b) = f_a(b)$, we finally arrive at the functional equation

$$f_a(b) = \langle \log(1+abz) \rangle + \left\langle f_a\left(\frac{a+bz}{1+abz}\right) - f_a(a) \right\rangle \tag{19}$$

which determines the required $f_a = f_a(a)$. A similar equation has been derived for the random-field Ising chain in [6] on the basis of preliminary considerations in [15]. The function $D(u)$, defined in [6, 7], is similar to our $f_a(b)$ but, in contrast to our formulation, the free energy is obtained by averaging $D(u)$ (with a field-dependent value of u) over the distribution of random fields. Therefore, we consider our formalism to be more suitable for the following studies in the neighbourhood of the point $a = 0$.

Redefining $x_n \to bx_n$ $(n = N-1, N)$ in (16*b*), (17) one easily shows that $f_a(b)$ is the function of ab and a/b only. Consequently, it holds that

$$f_a(b) = f_{-a}(-b)\,. \tag{20}$$

Definition (17) also implies that

$$f_a(0) = 0\,. \tag{21}$$

The supposed inequality $\langle \log z \rangle < 1$ leads to $f_0(b) = 0$.

Let us suppose that $\langle z^n \rangle < 1$ for all $n = 1, 2, \ldots$. With respect to (20), the Taylor expansion of $f_a(b)$ near $a, b = (0, 0)$ can be written as

$$f_a(b) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} a^n b^m \tag{22}$$

with $n + m =$ an even number. Substituting (22) into the functional equation (19) and comparing terms with the same powers in a and b we find in a systematic way the relations for c_{nm}. The coefficients c_{nm} with $n < m$ turn out to be equal zero. Equations for the first non-zero coefficients read

$$c_{11} = \langle z \rangle + c_{11}\langle z \rangle \tag{23a}$$

$$c_{22} = -\langle z^2 \rangle/2 - c_{11}\langle z^2 \rangle + c_{22}\langle z^2 \rangle \tag{23b}$$

$$c_{31} = -c_{11}\langle z \rangle + 2c_{22}\langle z \rangle + c_{31}\langle z \rangle \tag{23c}$$

etc. The resulting expansion of $f_a = f_a(a)$

$$f_a = \frac{\langle z \rangle}{1 - \langle z \rangle} a^2 - \frac{1}{2} \frac{\langle z^2 \rangle (1 + \langle z \rangle)^2 + 2\langle z \rangle^2 (1 - \langle z^2 \rangle)}{(1 - \langle z^2 \rangle)(1 - \langle z \rangle)^2} a^4 + \mathrm{O}(a^6) \tag{24}$$

coincides with that obtained in [9] for the random-field Ising chain.

The 'naive' expansion breaks down in the $2n$th order if $\langle z^n \rangle \geq 1$. This is not the case of the Ising chain with random interactions ($|z| < 1$) for which the expansion around $H^* = 0$ works in all orders of a^2. On the other hand, for the random-field chain with $z \in (0, \infty)$ the expansion around $J \to \infty$ can be highly non-trivial.

We can expand function $f_a(b)$, with a, b small and positive, also around $b = a$ in powers of $(b-a)$:

$$f_a(b) = f_a(a) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm} a^n (b-a)^m . \tag{25}$$

As a is small, we suppose that (25) converges for $|b-a| < a$. Then, coefficients associated with nth power of b in (22) and (25) should coincide. In this way we find $d_{11} = \langle z\rangle/(1-\langle z\rangle)$, and so it diverges for $\langle z\rangle \to 1^-$ in the same way as c_{11}.

These divergencies indicate the non-analyticity of $f_a(b)$ near $b = 0$ and on the line $a = b$:

$$f_a(b) \approx \tilde{c}_{11}(ab)^{\alpha^*} \tag{26a}$$

$$f_a(b) \approx f_a(a) + \tilde{d}_{11} a^{\beta^*} |b-a|^{\beta^*} \operatorname{sign}(b-a) . \tag{26b}$$

Setting in (26b) $b \to 0$ one obtains $0 = f_a(a) - \tilde{d}_{11} a^{2\beta^*}$, which yields, comparing with (26a), $\tilde{c}_{11} = \tilde{d}_{11}$, $\alpha^* = \beta^*$.

Substituting (26a) and (26b) into the functional equation (19) we get for the leading term the relation

$$\langle z^{\alpha^*}\rangle = 1 . \tag{27}$$

Thus, we have proven the non-analyticity of the free energy near $a \to 0$:

$$f_a(a) \approx a^{2\alpha^*} \tag{28}$$

with α^* given by relation (27). The same finding, together with an approximate form of the prefactor, has been found in [9] in a more complicated way and with the aid of different plausible (but not rigorously justified) arguments.

4. Naive expansion for quasi-one-dimensional systems

We can generalize our previous considerations from the naive expansion to the strip of width M. The varying field H_{nm} ($n = 1, \ldots, N$; $m = 1, \ldots, M$) of the Ising model studied here will be included in $z_{nm} = \exp(-2H_{nm})$ and constant nearest-neighbour bonds J in $a = \exp(-2J)$. It is evident that the statistical sum $Z^{(N)}$ consists of 2^M terms $Z_i^{(N)}$, $i = 0, 1, \ldots, L = 2^M - 1$. Each term corresponds to a given configuration of M spins in the Nth column. Let $Z_0^{(N)}$ ($Z_L^{(N)}$) correspond to the configuration 'all spins up (down)'. The recursive relation for $\tilde{Z}_i^{(N)} = Z_i^{(N)}/Z_0^{(N)}$ ($i \neq 0$) can easily be found:

$$\tilde{Z}_i^{(N)} = \Omega_i^{(N)} a^{\alpha_i} \frac{\tilde{Z}_i^{(N-1)} + a^{\beta_i} + \sum_{j\neq 0,i} a^{\gamma_{ij}} \tilde{Z}_j^{(N-1)}}{1 + \sum_{j\neq 0} a^{\beta_j} \tilde{Z}_j^{(N-1)}} \tag{29}$$

where in $\Omega_i = \Omega_i^{(N)} = \prod_{m=1}^{M\ \{i\}} z_{Nm}$ the product is over all sites on which spin is down. α_i is number of interaction lines which connect different spins in the ith configuration; for $0 < i < L$ it is positive, while

$$\alpha_L = 0 . \tag{30}$$

In equation (29) β_i is the number of spins 'down' and γ_{ij} number of bonds between sites $(N,k),(N-1,k)$, $k=1,\ldots,M$ which connect sites with different spins in respective ith and jth configurations. One easily finds that

$$|\beta_i-\beta_j|\le\gamma_{ij}\le\beta_i+\beta_j \qquad \gamma_{ii}=0\,. \tag{31}$$

Owing to (31), the numerator in (29) is proportional to a^{β_i} and so $\tilde{Z}_i^{(n)}=a^{\alpha_i+\beta_i}x_i^{(n)}$, $x_i^{(n)}\sim 1$ $(n=N-1,N)$. Thus we can write the recursive formula for $x_i^{(N)}$ as

$$x_i^{(N)}=\Omega_i^{(N)}\frac{1+\sum_{j=1}^{L}a^{\lambda_{ij}}x_j^{(N-1)}}{1+\sum_{j=1}^{L}a^{\eta_j}x_j^{(N-1)}} \tag{32}$$

with

$$\lambda_{ij}=\gamma_{ij}+\alpha_j+\beta_j-\beta_i\ge 0 \tag{33a}$$

$$\eta_i=\alpha_i+2\beta_i\ge 2\,. \tag{33b}$$

It is important to note that once i or j is smaller than L, $\lambda_{ij}>0$, while, owing to (30) and (31),

$$\lambda_{LL}=0\,. \tag{34}$$

By analogy with section 2 one can show that the quenched averaging of the free energy per site is equivalent to the problem of quenched averaging of $f_a=\lim_{N\to\infty}f_a^{(NM)}$

$$f_a^{(NM)}=\frac{1}{NM}\sum_{n=1}^{N}\log\Bigl(1+\sum_{i\neq 0}a^{\eta_i}x_i^{(n)}\Bigr) \tag{35}$$

for a given distribution $\rho(z)$ of independent z_{nm} (the auxiliary variables $x_i^{(n)}$ fulfil recursion (32) where N is changed by n). In the spirit of our strategy, we define

$$f_a^{(NM)}(b_1,\ldots,b_L)=\frac{1}{M}\int_{\mathcal{D}}\cdots\int_{\mathcal{D}}\mathrm{d}z_{11}\cdots\mathrm{d}z_{NM}\,\rho(z_{11})\cdots\rho(z_{NM})$$
$$\times\log\Bigl(1+\sum_{i\neq 0}b_i a^{\eta_i-1}x_i^{(N)}\Bigr)\,. \tag{36}$$

Thus, $f_a=\lim_{N\to\infty}f_a^{(NM)}(a,\ldots,a)$. Using recursive relation (32) we obtain in the limit $N\to\infty$ the functional equation

$$f_a(b_1,\ldots,b_L)=\frac{1}{M}\Bigl\langle\log\Bigl(1+\sum_{i\neq 0}b_i a^{\eta_i-1}\Omega_i\Bigr)\Bigr\rangle+\bigl(f_a(\tilde{b}_1,\ldots,\tilde{b}_L)-f_a(a,\ldots,a)\bigr) \tag{37a}$$

with $\langle\cdots\rangle=\int\mathrm{d}z_1\cdots\mathrm{d}z_M\,\rho(z_1)\cdots\rho(z_M)\cdots$, and

$$\tilde{b}_i a^{\eta_i-1}=\frac{a^{\eta_i}+\sum_{j\neq 0}a^{\lambda_{ji}}b_j a^{\eta_j-1}\Omega_j}{1+\sum_{j\neq 0}b_j a^{\eta_j-1}\Omega_j} \tag{37b}$$

which determines the required $f_a = f_a(a, \ldots, a)$.

Let us suppose that the naive expansion of $f_a(b_1, \ldots, b_L)$ exists. Then, in the limit $a \to 0$, $b \to 0$,

$$Mf_a(b_1, \ldots, b_L) = \sum_{i \neq 0} C_i b_i a^{\eta_i - 1} + \cdots . \tag{38}$$

From (37*a*) and (37*b*) we can establish equations for coefficients C_i. If exponent $\lambda_{ii} > 0$, then (37*a*) and (37*b*) yield the trivial relation for C_i:

$$C_i = \langle \Omega_i \rangle \qquad (= \langle z \rangle^{\beta_i}) . \tag{39a}$$

For $i = L$, however, owing to (34), we obtain

$$C_L = \frac{\langle \Omega_L \rangle}{1 - \langle \Omega_L \rangle} \tag{39b}$$

where $\langle \Omega_L \rangle = \langle z \rangle^M$ and so C_L diverges for $\langle z \rangle = 1$ as in the 1D case. As $\eta_L = 2M$, the non-analyticity of the free energy appears first in the $2M$th order of expansion in a.

It is still necessary to prove that the non-analyticity of $f_a(a, \ldots, a)$ in the $2M$th order does not occur in the next terms of the expansion. Using (37*a*) and (37*b*) we derive

$$Mf_a(b_1, \ldots, b_L) = \sum_{i \neq 0} C_i b_i a^{\eta_i - 1} + \sum_{\substack{ij \neq 0 \\ \text{except } i=j=L}} C_{ij} a^{\lambda_{ij}} b_i a^{\eta_i - 1} + \cdots$$
$$+ \sum_{ij \neq 0} D_{ij} b_i a^{\eta_i - 1} b_j a^{\eta_j - 1} + \cdots . \tag{40}$$

Here $C_{ij} = 0$ once $\beta_i + \beta_j = \gamma_{ij}$ (there are no bonds between the ith and jth configurations which connect two spins 'down'), otherwise

$$C_{ij} = \langle \Omega_i \rangle \langle \Omega_j \rangle \qquad i, j \neq L \tag{41a}$$

$$C_{iL} = C_{Li} = \frac{\langle \Omega_i \rangle \langle \Omega_L \rangle}{1 - \langle \Omega_L \rangle} \qquad i \neq L . \tag{41b}$$

The symmetric coefficients D_{ij} are given by

$$D_{ij} = -\frac{1}{2} \langle \Omega_i \Omega_j \rangle \qquad i, j \neq L \tag{42a}$$

$$D_{iL} = -\frac{1}{2} \frac{\langle \Omega_i \Omega_L \rangle}{1 - \langle \Omega_L \rangle} \qquad i \neq L \tag{42b}$$

$$D_{LL} = -\frac{1}{2} \frac{\langle \Omega_L^2 \rangle (1 + \langle \Omega_L \rangle)}{(1 - \langle \Omega_L \rangle)(1 - \langle \Omega_L^2 \rangle)} \tag{42c}$$

where $\langle \Omega_L^2 \rangle = \langle z^2 \rangle^M$. The construction of the expansion was based on the inequality

$$\lambda_{ij} \leq \eta_j \tag{43}$$

(the equality holds if $\beta_i + \beta_j = \gamma_{ij}$), which enabled us to order terms according to their powers. On account of the identity

$$\lambda_{ij} + \eta_i = \lambda_{ji} + \eta_j \tag{44}$$

terms with coefficients C_{ij} and C_{ji} contain the same power of a. We see that the next failure of the naive expansion of $f_a(a, \ldots, a)$ occurs in orders higher than $2M$. Consequently, the assumption of the non-analyticity of $f_a(a, \ldots, a)$ in the $2M$th order is correct.

As M increases to infinity, the free energy of the corresponding two-dimensional system is analytical in all finite orders of a.

5. Non-analyticity of an Ising chain with random bonds

The naive expansion of the free energy of the random-field Ising chain can fail near $a \to 0$ which corresponds to infinite constant bonds $J \to \infty$. We now study a counterpart in Ising chains with random interactions (the fraction of which is infinitely large) and investigate the analytical properties of its free energy.

The random bonds of the chosen model take 0, $+J^*$, $-J^*$ with respective probabilities p_0, p_+, p_- constrained by $p_0 + p_+ + p_- = 1$. Then we have

$$\rho(z) = p_0\delta(z) + p_+\delta(z - K) + p_-\delta(z + K) \tag{45}$$

with $K = \tanh J^*$. Equation (19) reads

$$f_a(b, K) = p_+ \log(1 + abK) + p_- \log(1 - abK) - (p_+ + p_-) f_a(a, K) + p_+ f_a\left(\frac{a + bK}{1 + abK}, K\right) + p_- f_a\left(\frac{a - bK}{1 - abK}, K\right) \tag{46}$$

with $a = \tanh(H^*)$ (the uniform field H^* is, for simplicity, positive). For $K = 1$ ($J^* \to \infty$) this equation has a more suitable form:

$$f_a(b, 1) = p_+ \log(1 + ab) + p_- \log(1 - ab) - (p_+ + p_-) f_a(a, 1) + p_+ f_a\left(\frac{a + b}{1 + ab}, 1\right) + p_- f_a\left(\frac{a - b}{1 - ab}, 1\right) \tag{46'}$$

and can be solved analytically, using the *ansatz*

$$f_a(b, 1) = \sum_{n=-\infty}^{\infty} \alpha_n \log[1 + b g_n(a)] . \tag{47}$$

The functions $g_n(a)$, $g_{-n}(a) = -g_n(a)$ satisfy the recursive relations

$$g_{n+1}(a) = \frac{a + g_n(a)}{1 + a g_n(a)} . \tag{48}$$

Evidently, $g_n(a) = \tanh(nH^*)$. For coefficients α_n, constrained by $\sum \alpha_n = 1$, we obtain the system of linear equations

$$\alpha_n = p_0\delta_{n,0} + p_+\alpha_{n-1} + p_-\alpha_{-n-1} \tag{49}$$

which gives

$$\alpha_n = \frac{p_0}{\sqrt{(1+p_+)^2 - p_-^2}\sqrt{(1-p_+)^2 - p_-^2}} \{\mu^{|n|} + (p_- - p_+)\mu^{|n+1|}\} \tag{50a}$$

with

$$\mu = \frac{\sqrt{(1+p_+)^2 - p_-^2} - \sqrt{(1-p_+)^2 - p_-^2}}{\sqrt{(1+p_+)^2 - p_-^2} + \sqrt{(1-p_+)^2 - p_-^2}}. \tag{50b}$$

In what follows we will deal with the special case $p_+ = p_- = p$ ($< 1/2$). Equations (50a) and (50b) then read

$$\alpha_n = C\mu^{|n|} \tag{51a}$$

where $\mu < 1$ is given by

$$1/2p = \cosh(\log \mu) \tag{51b}$$

and

$$C = (1-\mu)/(1+\mu). \tag{51c}$$

This solution can be obtained also from 'classical' method of Derrida: by solving the functional equation for the probability density $P(x)$ (in a manner similar to that presented here) one obtains that $P(x)$ consists of an infinite number of weighted δ-function peaks localized at integers multiplied by H^*. This is due to the fact that recursive formula (16b) starting from 0, always gives the same succession of xs. The same effect is visible from our equation (46′): for $a = \tanh(H^*)$, $b = \tanh(y)$, $g_n(a) = \tanh(nH^*)$, the y are confined to values nH^*, n integer, and so the values of b, which appear in (46′) are confined to the simple countable set.

For $K < 1$, this 'ideal' picture will be damaged. The probability density $P(x)$ is expected to be of a considerably different form: the δ-function peaks will move and, in the limit $N \to \infty$, we should expect $P(x)$ of the form of some broaden peaks rather than of an infinite number of δ-peaks. In our picture, one can no longer consider $f_a(b, K)$ in the form of the infinite sum. We have not found a method of solving this problem exactly. Nevertheless, we have succeeded in analysing of the behaviour of $f_a(b, 1-\varepsilon)$ in the limit $\varepsilon \to 0$.

Before proceeding further, let us rewrite $f_a(b, 1)$ into the more suitable form:

$$f_a(b,1) = C \sum_{n>0} \mu^n \log\left(1 - b^2 g_n^2(a)\right). \tag{52}$$

$f_a(b, 1)$ converges to a finite constant for any $0 \le b \le 1$. However, this is not true for its derivative

$$\frac{\partial f_a(b,1)}{\partial b} = -2bC \sum_{n>0} \mu^n \frac{g_n^2(a)}{1 - b^2 g_n^2(a)}. \tag{53}$$

Indeed, one can easily find that for $b = 1$ it converges only for

$$\alpha^* = -\frac{\log(\mu)}{2H^*} > 1. \tag{54}$$

For this case, we can differentiate equation (46) with respect to K and seek

$$h_a(b) = f_a(b,1) - f_a(b,1-\varepsilon) \tag{55}$$

in the standard way as

$$h_a(b) = \frac{\partial f_a(b,K)}{\partial K}\bigg|_{K=1} \varepsilon .$$

For $0 < \alpha^* < 1$, the sum (53) diverges for $b = 1$, and so $f_a(b,1)$ becomes non-analytical in the neighbourhood of $b = 1$. It can be shown, using for instance the procedure presented in appendix 1 of [9], that

$$f_a(b,1) = f_a(1,1) - \mathrm{O}\left[(1-b)^{\alpha^*}\right] \qquad b \to 1 . \tag{56}$$

One can also estimate the exponent α^* by an another method. Taking $b = 1 - \tilde{\varepsilon}$ ($\tilde{\varepsilon} \to 0$) we have for $a < 1$ $(a+b)/(1+ab) \approx 1 - [(1-a)/(1+a)]\tilde{\varepsilon}$, $(a-b)/(1-ab) \approx -1 + [(1+a)/(1-a)]\tilde{\varepsilon}$. Inserting this into (46′) and considering $f_a(b,1) = f_a(-b,1)$ we get for the terms $\sim \tilde{\varepsilon}^{\alpha^*}$ an equation

$$\frac{1}{p} = \left(\frac{1-a}{1+a}\right)^{\alpha^*} + \left(\frac{1+a}{1-a}\right)^{\alpha^*} = 2\cosh(-2H^*\alpha^*) . \tag{57}$$

Owing to (51b) we arrive at (56).

The functional equation for $h_a(b)$ can be now derived by the substraction of equations (46′) and (46) with $K = 1 - \varepsilon$. The difference

$$f_a\left(\frac{a+b}{1+ab},1\right) - f_a\left(\frac{a+b(1-\varepsilon)}{1+ab(1-\varepsilon)},1-\varepsilon\right)$$

could be written as

$$h_a\left(\frac{a+b(1-\varepsilon)}{1+ab(1-\varepsilon)}\right) + \Delta_+$$

with

$$\Delta_+ = f_a\left(\frac{a+b}{1+ab},1\right) - f_a\left(\frac{a+b(1-\varepsilon)}{1+ab(1-\varepsilon)},1\right) . \tag{58}$$

Owing to (56), Δ_+ is of order ε^{α^*} for $b \to 1$ (e.g. $b = 1 - \lambda\varepsilon$). In this way we obtain the system of equations

$$\frac{1}{p}h_a(b) = -\frac{2a^2b^2\varepsilon}{1-a^2b^2} + \Delta_+ + \Delta_- - 2h_a(a) + h_a\left(\frac{a+b(1-\varepsilon)}{1+ab(1-\varepsilon)}\right) + h_a\left(\frac{a-b(1-\varepsilon)}{1-ab(1-\varepsilon)}\right) . \tag{59}$$

We can rewrite functional equation (59) as a system of linear equations by setting $b \to i/N$, $N \gg 1/\varepsilon$, $i = 1,\ldots,N$:

$$\sum_{j=1}^{N} A_{ij}h_j = c_i^{(1)}\varepsilon^{\alpha^*} + c_i^{(2)}\varepsilon + \cdots \tag{60}$$

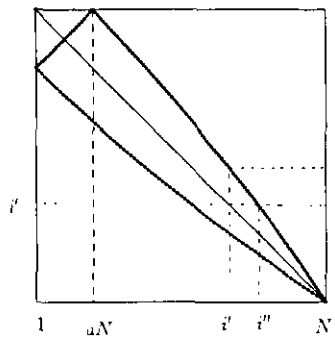

Figure 2. The form of the matrix A from equation (60). The diagonal full line represents $A_{ii} = 1/p$, and the vertical broken line represents $A_{ji} = 2$ with $j = aN$. The heavy full curves determine the position of $\tilde{h}_a((a + b(1-\varepsilon))/(1 + ab(1-\varepsilon)))$ and $\tilde{h}_a((a - b(1-\varepsilon))/(1 - ab(1-\varepsilon)))$, respectively. It is clear from the structure of equations, that if $\tilde{h}_i = 0$ for all $i < i'$, then $\tilde{h}_i = 0$ also for $i' < i < i''$. By induction, one obtains $\tilde{h}_i = 0$ for all $i < N$.

where the structure of matrix A is depicted in figure 2 and components of vector $c^{(1)}$ are non-zero only for large values of i . Omitting the term $\mathrm{O}(\varepsilon)$ in (60), one gets the system of linear equations, which has the unique solution $\tilde{h}_i$, proportional to ε^{α^*}. Let us suppose for the moment, that $\tilde{h}_i = 0$ for all i smaller than some i'. Then one easily finds, from the structure of equations (60), that $\tilde{h}_i = 0$ also for all $i' < i < i''$ (figure 2). The same procedure can be repeated, and, finally, one obtains *all* $\tilde{h}_i = 0$, except, maybe $i = N$. This, however, disagrees with the fact that vector $c^{(1)}$ has *non-zero* components for large i.

On the basis of the above considerations we conclude that

$$h_a(b) \sim \varepsilon^{\alpha^*} \tag{61}$$

for all b, and so

$$f_a(a, 1-\varepsilon) \approx f_a(a,1) - \mathrm{O}(\varepsilon^{\alpha^*}). \tag{62}$$

We can proceed further and analyse the higher-order terms of the expansion in ε. In this way, we need the kth derivative $f_a^{(k)}(b,1) = \partial^k f_a(b,1)/\partial b^k$. One easily finds, using the same considerations as explained above, that the condition

$$\mu \exp(2kH^*) < 1 \tag{63}$$

is necessary to assure the convergence of $f_a^{(k)}(b,1)$ for $b = 1$. Thus, if

$$k < \alpha^*(a,p) < k+1 \tag{64}$$

then the expansion of $f_a(b,1)$ near $b = 1$ becomes non-analytical in the $(k+1)$th order in $(1-b)$. We therefore conclude, on the base of the same arguments as above, that the inequality (64) determines parameters a, p for which $f_a(a, 1-\varepsilon)$ is analytical up to the kth order in ε, and is singular in the $(k+1)$th order as ε^{α^*}.

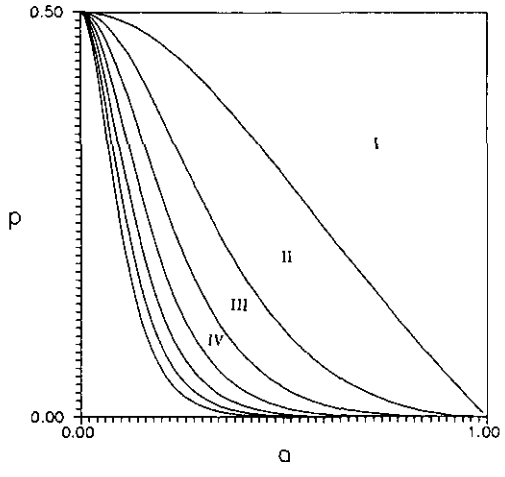

Figure 3. The regions in the (a,p)-plane in which $f_a(a)$ is non-analytical in the first (I), second (II), ... order of expansion in ε.

6. Concluding remarks

We would like to emphasize that the non-analyticity of the free energy in the regime with 'a fraction of almost infinite random interactions' originates from the singular behaviour of function $f_a(b, K = 1)$ near $b = 1$. We therefore conclude that our approach is so far the only one able to explain this interesting phenomena. The singular behaviour of f_a at a given point on plane $(\tanh H^*, p)$ (figure 3) is determined by the positive parameter α^* defined in (54). Its integer part $[\alpha^*]$ implies the last order of ε at which the Taylor expansion does not break down. It is clear that for *each* point on the plane $(\tanh H^*, p)$ the Taylor expansion breaks down at a certain order. Since for an arbitrary small field H^* and an arbitrary small concentration p of infinite plus/minus interactions a spin configuration cannot satisfy all interactions and the field simultaneously, this fact confirms the effect of the frustration on the singular properties of the free energy.

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